ENTANGLEMENT ENTROPY ON THE FUZZY SPHERE

Asato Tsuchiya

Shizuoka Univ.

In collaboration with S. Okuno and M. Suzuki (Shizuoka Univ.)

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Introduction

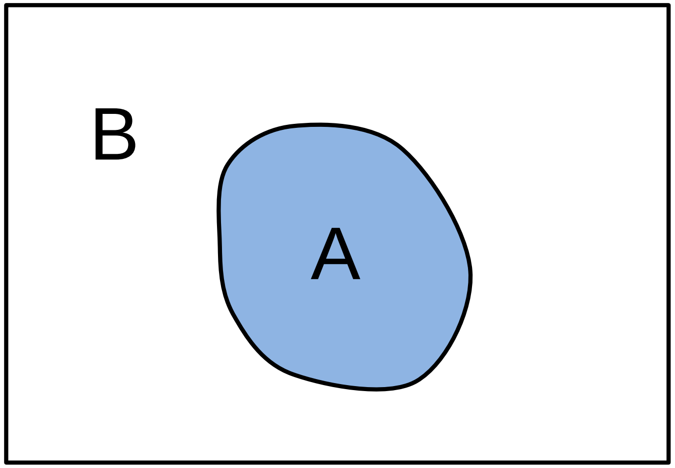
Entanglement entropy

- Definition of EE

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\rho_A = \mathrm{Tr}_B[\rho_{tot}]$$

$$\boxed{S_A = -\mathrm{Tr}[\rho_A \log \rho_A]}$$



- EE in local field theories

$$S_A \propto \frac{|\partial A|}{\epsilon^{d-1}}$$

$|\partial A|$: area of boundary

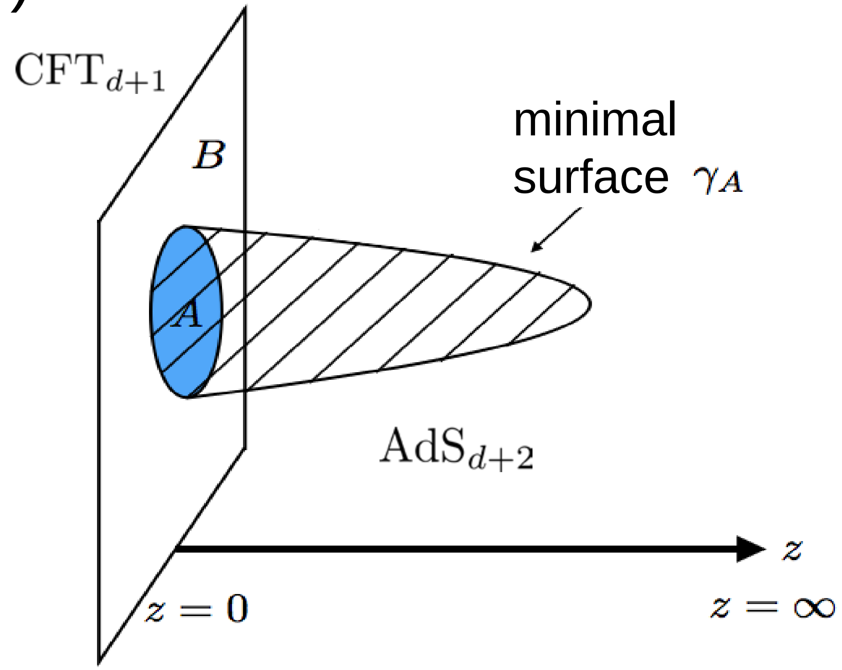
ϵ : UV cutoff

d : dimension of space

Quantum entanglement and geometry

Ryu-Takayanagi formula ('06)

$$S_A = \frac{Area(\gamma_A)}{4G_N}$$



cf.) Hung's talk
Chen's talk

1st law of thermodynamics → Einstein eq.

Noncommutative plane

- noncommuative plane

$$[\hat{x}_1, \hat{x}_2] = i\theta \qquad\qquad \theta \text{ : real}$$

simplest example of noncommutative space

- conjugate momenta

$$\hat{p}_1 = \theta^{-1}\hat{x}_2, \;\; \hat{p}_2 = -\theta^{-1}\hat{x}_1 \qquad [\hat{x}_i, \hat{p}_j] = i\delta_{ij} \;\; (i,j=1,2)$$

coordinates and momenta are not independent

- coherent state

$$\hat{a} = \hat{x}_1 + i\hat{x}_2 \qquad \hat{a}^\dagger = \hat{x}_1 - i\hat{x}_2 \quad \Rightarrow \quad [\hat{a}, \hat{a}^\dagger] = 2\theta$$

$$|z\rangle = e^{-\frac{|z|^2}{4\theta}} e^{\frac{z}{2\theta}\hat{a}^\dagger}|0\rangle \quad \Rightarrow \quad \hat{a}|z\rangle = z|z\rangle$$

Berezin symbol

- Berezin symbol

Cf.) Iso-Kawai-Kitazawa ('00)

$$f_{\hat{A}}(z,z^*) = f_{\hat{A}}(x) = \langle z|\hat{A}|z\rangle$$

- derivative

$$f_{[\hat{p}_i,\hat{A}]}(x) = -i\partial_{x_i} f_{\hat{A}}(x)$$

- product

$$\begin{aligned} f_{\hat{A}}(x) \star f_{\hat{B}}(x) &\equiv f_{\hat{A}\hat{B}}(x) \\ &= e^{-\frac{i\theta}{2}(\partial_{x_1}\partial_{y_2}-\partial_{x_2}\partial_{y_1})} f_{\hat{A}}(x) f_{\hat{B}}(y)|_{x=y} \\ &= f_{\hat{A}}(x) f_{\hat{B}}(x) - \frac{i\theta}{2}(\partial_{x_1} f_{\hat{A}}(x)\partial_{x_2} f_{\hat{B}}(x) - \partial_{x_2} f_{\hat{A}}(x)\partial_{x_1} f_{\hat{B}}(x)) + \cdots \end{aligned}$$

- trace

$$\mathrm{tr}(\hat{A}) = \int \frac{d^2x}{2\pi\theta} f_{\hat{A}}(x)$$

Moyal product
non-local

Field theory on noncommutative plane

- a matrix model with infinite matrix size

$$S = 2\pi\theta \mathrm{tr}\left(-\frac{1}{2}[\hat{p}_i, \hat{\phi}]^2 + \frac{m^2}{2}\hat{\phi}^2 + \frac{\lambda}{4}\hat{\phi}^2 \right)$$

- correspondence between matrix and field

$$\phi(x) = f_{\hat{\phi}}(x) = \langle z|\hat{\phi}|z\rangle \qquad z = x_1 + ix_2$$

- scalar field theory on noncommutative plane

$$S = \int d^2x \left(\frac{1}{2}(\partial_{x_i}\phi(x))^2 + \frac{m^2}{2}\phi(x)^2 + \underline{\frac{\lambda}{4}\phi(x)\star\phi(x)\star\phi(x)\star\phi(x)} \right)$$

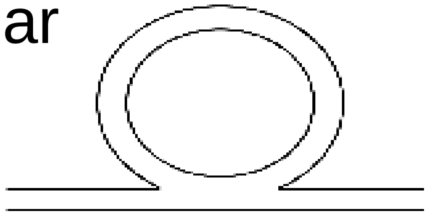
effect of noncommutavity appears only in interaction term

UV/IR mixing

Minwalla-Raamsdonk-Seiberg ('99)

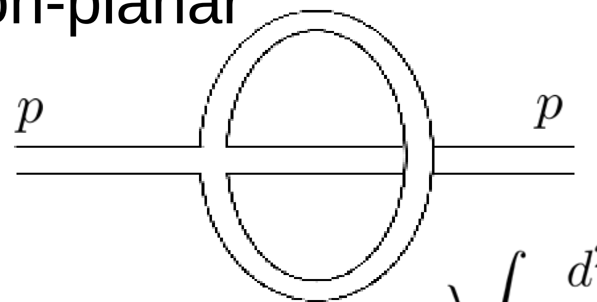
Ex.) 1-loop correction to propagator

planar



$$-2\lambda \int \frac{d^2 q}{(2\pi)^2} \frac{1}{q^2 + m^2}$$

non-planar



$$-\lambda \int \frac{d^2 q}{(2\pi)^2} \frac{e^{-i\theta(p_1 q_2 - p_2 q_1)}}{q^2 + m^2}$$

$$= -\frac{\lambda}{2\pi}\left(\gamma + \log\left(\frac{m}{2}\sqrt{\theta^2 p^2 + \frac{1}{\Lambda^2}}\right)\right)$$

Λ : UV cutoff

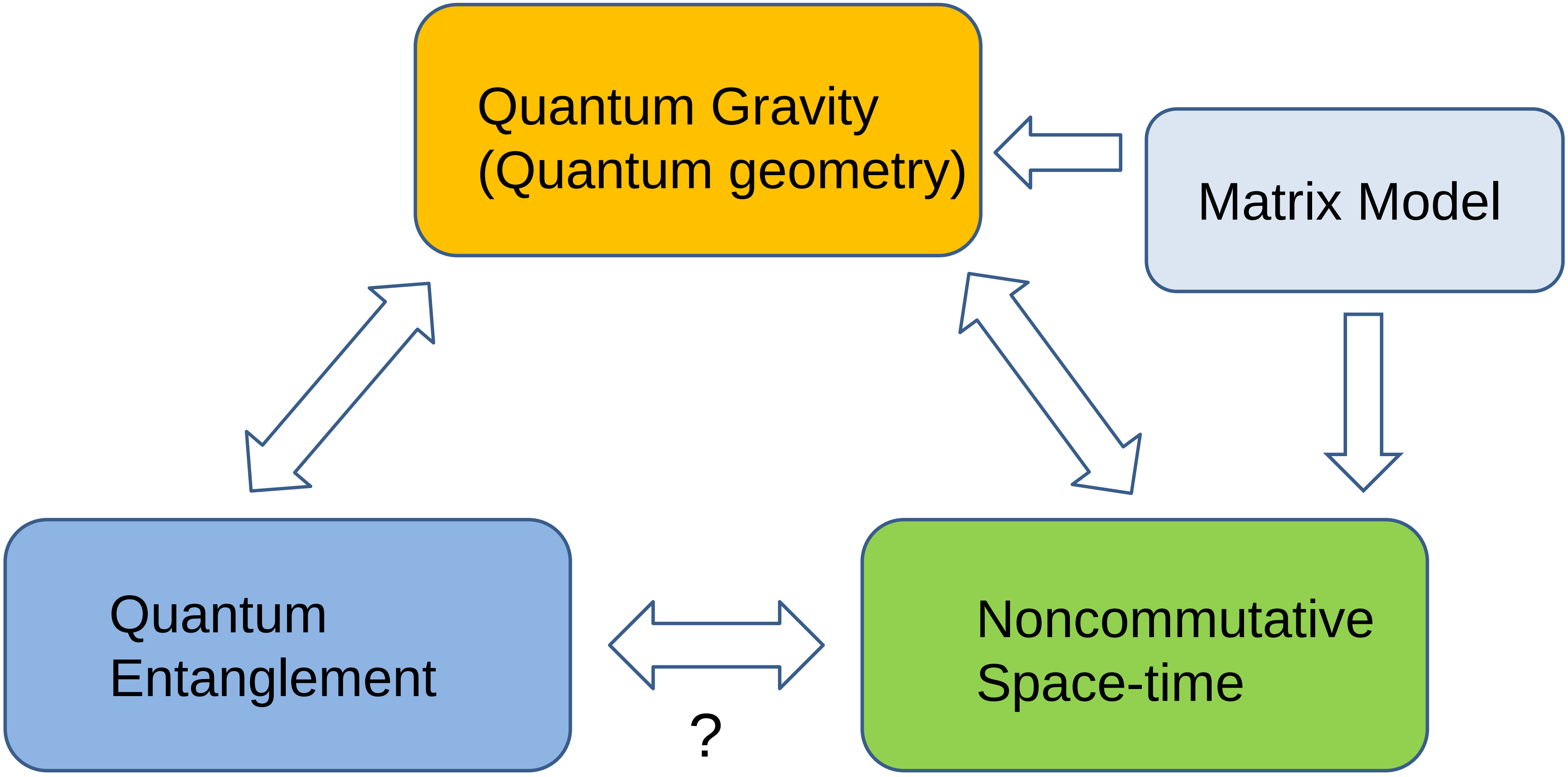
$\theta = 0 \quad \Lambda \to \infty$ ➡ logarithmic div. ~ ordinary field theory

$\theta \neq 0 \quad \Lambda \to \infty$ ➡ $\log\left(\frac{m}{2}\sqrt{\theta^2 p^2}\right) \qquad p \to 0$ IR div.

$r \sim 1/p \sim \theta\Lambda$

UV/IR mixing

Motivation



Quantum Gravity
(Quantum geometry)
Matrix Model
Quantum
Entanglement
?
Noncommutative
Space-time

What we will study

- calculate EE in scalar field theory on the fuzzy sphere using a different method from the method in the previous work

  Karczmarek-Sabella-Garnier ('13)
  Sabella-Garnier ('14)

  ⇨ applicable to free theory at zero temperature

  no restriction for our method

- study interacting theory in which non-commutativity should contribute

  'volume law' ?

  Cf.) EE in a non-local theory

  Shiba-Takayanagi ('13)

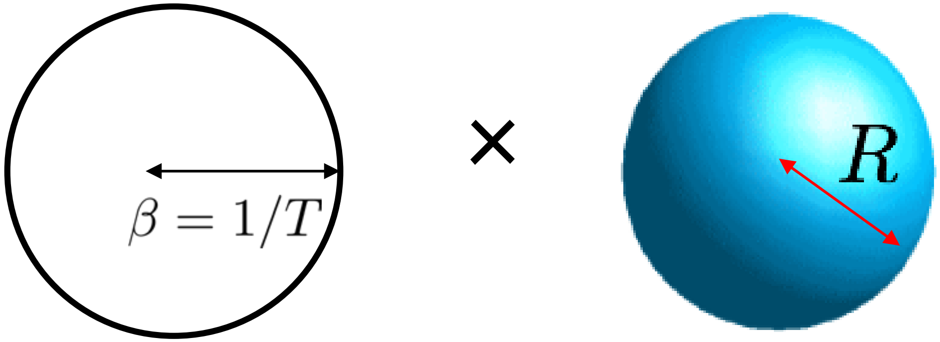
- finite temperature effect ?

Contents

Scalar field theory on the fuzzy sphere

Scalar field theory on S$^1\times$S^2

$$S_C = \frac{R^2}{4\pi}\int_0^\beta dt \int d\Omega \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2R^2}(\mathcal{L}_i\phi)^2 + \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4\right)$$



$\mathcal{L}_i \quad (i = 1, 2, 3)$: angular momentum operators

Scalar field theory on the fuzzy sphere

matrix quantum mechanics

$$S_{NC} = \frac{R^2}{2j+1} \int_0^\beta dt\ \mathrm{tr} \left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2R^2}[L_i, \Phi]^2 + \frac{\mu^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4 \right)$$

$\Phi(t)$: $(2j+1) \times (2j+1)$ Hermitian matrix

L_i : generators of spin j rep. of SU(2)

$$[L_i, L_j] = i\epsilon_{ijk} L_k$$

can be put on a computer

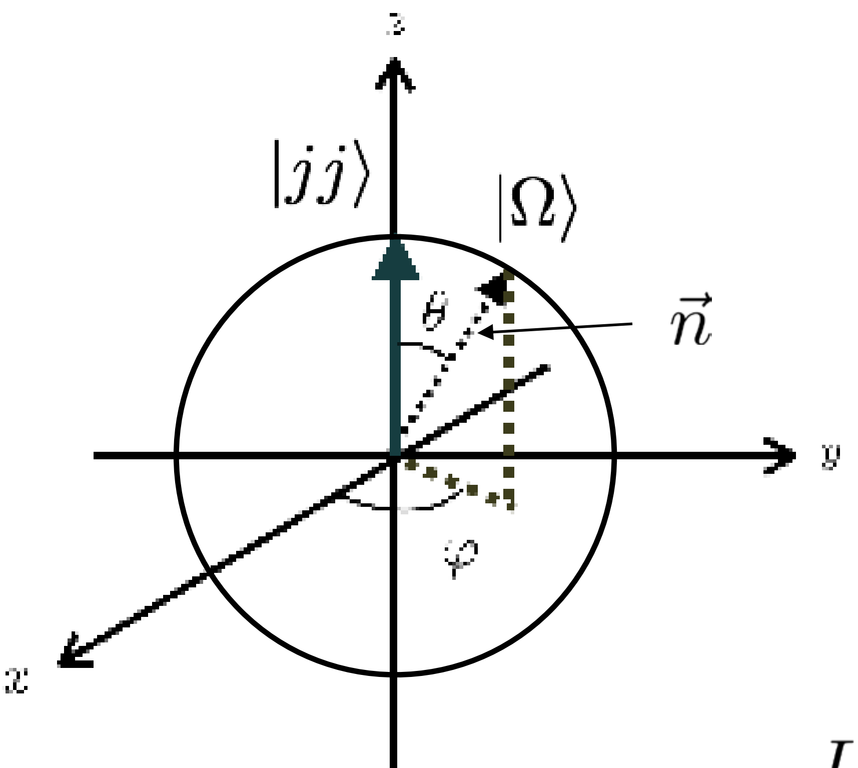
$j \to \infty$ j : UV cutoff

$$S_C = \frac{R^2}{4\pi} \int_0^\beta dt \int d\Omega \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2R^2}(\mathcal{L}_i\phi)^2 + \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \right)$$

only at tree level

Bloch coherent state

Gazeau et al.(09)



L_i : generators of spin j rep. of SU(2)

$$L_\pm = L_1 \pm iL_2$$

$$L_\pm|jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|jm \pm 1\rangle$$

$$L_3|jm\rangle = m|jm\rangle$$

$$\Omega = (\theta, \varphi)$$

$$|\Omega\rangle = e^{i\theta(\sin\varphi L_1 - \cos\varphi L_2)}|jj\rangle$$

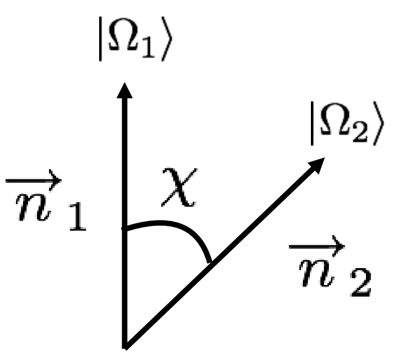
$$L_3|jj\rangle = j|jj\rangle \Rightarrow \vec{n}\cdot\vec{L}|\Omega\rangle = j|\Omega\rangle$$

Properties of Bloch coherent state

- $\sum_i (\Delta L_i)^2$ is minimum
- completeness

$$\frac{2j+1}{4\pi}\int d\Omega|\Omega\rangle\langle\Omega| = \sum_{j=-m}^{m} |jm\rangle\langle jm| = 1$$

- inner product



$$|\langle\Omega_1|\Omega_2\rangle| = \left(\cos\frac{\chi}{2}\right)^{2j}$$

$$\chi = \frac{2}{\sqrt{j}} \quad \Rightarrow \quad \left(\cos\frac{\chi}{2}\right)^{2j} \approx \left(1-\frac{1}{2j}\right)^{2j} \approx e^{-1}$$

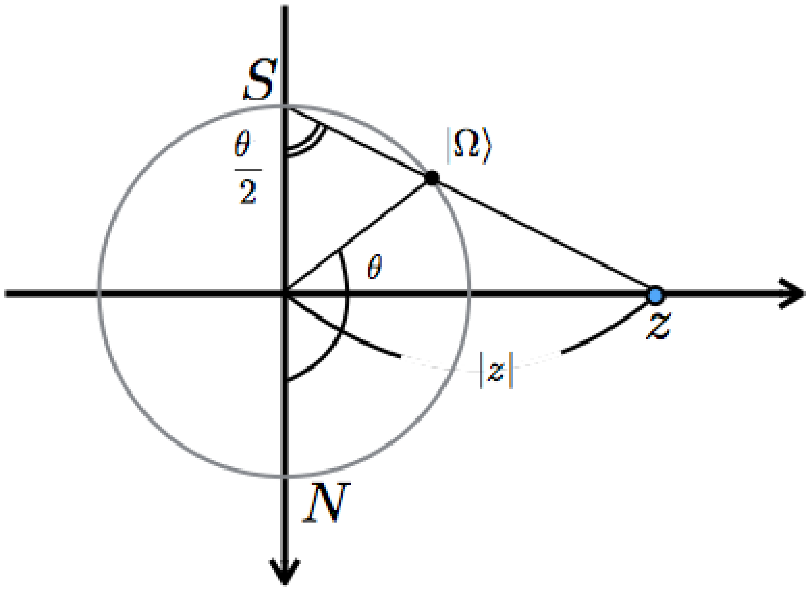
width of wave packet ~ $1/\sqrt{j}$

Berezin symbol

- Berezin symbol

$$f_A(\Omega) = \langle \Omega | A | \Omega \rangle$$

$$\Rightarrow \quad f_{[L_i, A]}(\Omega) = \mathcal{L}_i f_A(\Omega)$$



- star product

$$f_A(\Omega) \star f_B(\Omega) \equiv f_{AB}(\Omega)$$

$$\xRightarrow[j \to \infty]{} \quad R^2 \int \frac{d\Omega}{4\pi} f_A(\Omega) \star f_B(\Omega) = R^2 \int \frac{d\Omega}{4\pi} f_A(\Omega) f_B(\Omega)$$

- stereographic projection

$$\theta \sim R^2/(2j)$$

$$z = R \tan\frac{\theta}{2} e^{i\varphi} \quad \Rightarrow \quad f_A \star f_B(0,0) \approx e^{\frac{1}{2j}\frac{\partial}{\partial z}\frac{\partial}{\partial w}} f_A(z,\bar{z}) f_B(w,\bar{w})|_{z=w=0}$$

Matrix vs Field

- correspondence between matrix and field

$$\boxed{\phi(t,\Omega) = f_{\Phi(t)} = \langle \Omega | \Phi(t) | \Omega \rangle}$$

$$S_{NC} = \frac{R^2}{2j+1} \int_0^\beta dt \ \mathrm{tr} \left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2R^2}[L_i, \Phi]^2 + \frac{\mu^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4 \right)$$

$$\Updownarrow \quad j \to \infty$$

$$S_C = \frac{R^2}{4\pi} \int_0^\beta dt \int d\Omega \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2R^2}(\mathcal{L}_i \phi)^2 + \frac{\mu^2}{2}\phi^2 + \boxed{\frac{\lambda}{4}\phi_\star^4} \right)$$

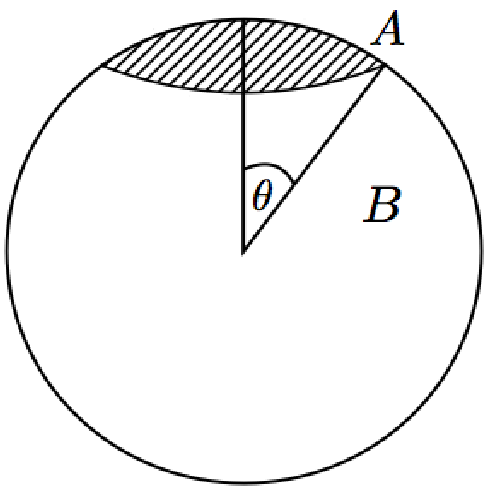
$\Rightarrow$ UV/IR anomaly

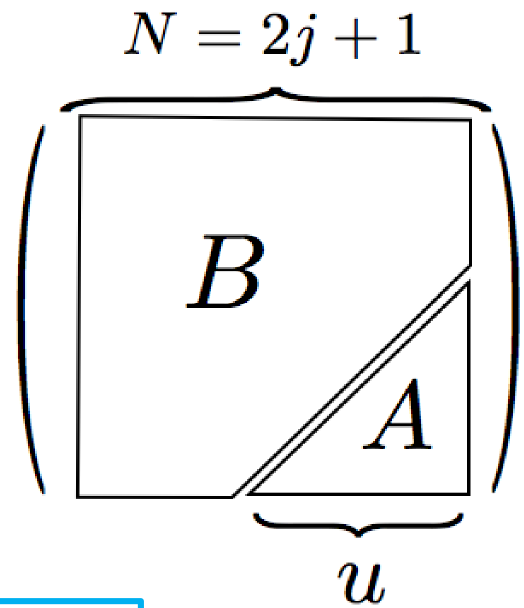
Chu-Madore-Steinacker ('01)

$$r \sim \theta\Lambda \sim \frac{R^2}{2j} \times \frac{2j}{R} \sim R$$

Calculation of EE

Division of the fuzzy sphere and matrix





$$\boxed{x = 1 - \cos\theta = \frac{u}{2j}} \quad \propto \text{ volume of A}$$

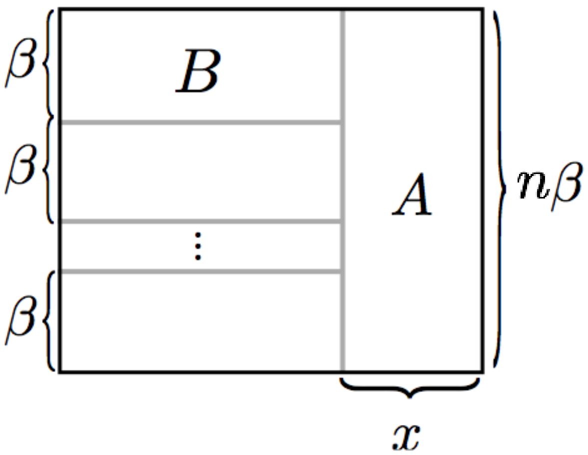
Karczmarek-Sabella-Garnier ('13)

$$f_\Phi(\Omega) = \sum_{m,m'} \langle \Omega | jm \rangle \langle jm | \Phi | jm' \rangle \ \langle jm' | \Omega \rangle$$

$$\langle \Omega | jm \rangle \langle jm' | \Omega \rangle \sim \left(\cos\frac{\theta}{2}\right)^{2j+m+m'} \left(\sin\frac{\theta}{2}\right)^{2j-m-m'} e^{i(m-m')\varphi}$$

Replica Method

- Boundary condition for replicas



Region A $\quad \Phi_I(\beta) = \Phi_{I+1}(0)$

Region B $\quad \Phi_I(\beta) = \Phi_I(0)$

$$I = 1, \cdots, n$$

$$\Rightarrow \quad \mathrm{Tr}\rho_A^n = \frac{Z(x,n)}{Z^n}$$

- Replica method

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A = -\lim_{n\to 1} \frac{\partial}{\partial n} \ln \mathrm{Tr}\rho_A^n = -\lim_{n\to 1} \frac{\partial}{\partial n} \ln \left(\frac{Z(x,n)}{Z^n} \right)$$

Our method of calculation

- quantity we calculate $x = 1 - \cos\theta$

$$\frac{\partial S_A(x)}{\partial x} = \frac{\partial}{\partial x}\left[-\lim_{n\to 1}\frac{\partial}{\partial n}\ln\left(\frac{Z(x,n)}{Z^n}\right)\right] = \lim_{n\to 1}\frac{\partial}{\partial x}\frac{\partial}{\partial n}F[x,n]$$

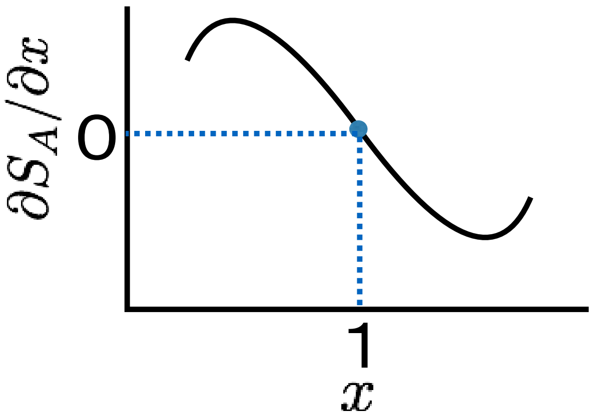
$$\to \frac{\partial}{\partial x}(F[x,n=2] - F[x,n=1]) = \lim_{\substack{j\to\infty \\ (\epsilon\to 0)}}\frac{F[x+\varepsilon,n=2] - F[x,n=2]}{\varepsilon}$$

$$\epsilon = \frac{1}{2j}$$

- property of x-derivative of in the $\beta \to \infty$ limit

$$S_A = S_B \Rightarrow S_A(x) = S_A(2-x)$$

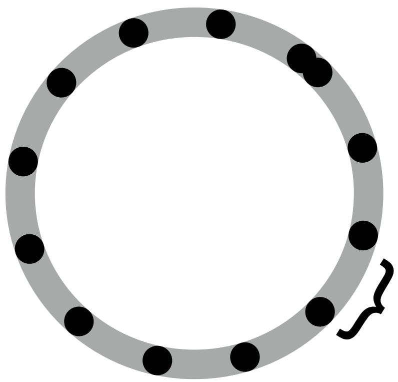
$$\frac{\partial S_A}{\partial x}(x) = -\frac{\partial S_A}{\partial x}(2-x)$$



Our method of calculation (cont'd)

- discretization of time direction

S^1



a lattice spacing

- free theory $\lambda = 0$

$$S_{NC} = \sum_{n,l,m_1,m_2,m_3,m_4} \Phi^*_{m_1 m_2}(n) T_{nm_1m_2,lm_3m_4} \Phi_{m_3m_4}(l)$$

$$\Rightarrow \quad F[x, \alpha = 2] = \frac{1}{2} \log \det T + \text{const.}$$

we calculate detT directly numerically

Our method of calculation (cont'd)

- interacting theory

$$F[x+\varepsilon, n=2] - F[x, n=2] = \boxed{\int_0^1 d\gamma} \boxed{\langle S_{x+\varepsilon} - S_x \rangle_\gamma}$$

MC simulation

Simpson formula

interpolating action

$$S_{int} = (1-\gamma)S_{x+\varepsilon} + \gamma S_x$$

$$(0 \leq \gamma \leq 1)$$

γ 0.1 step

Results for free theory

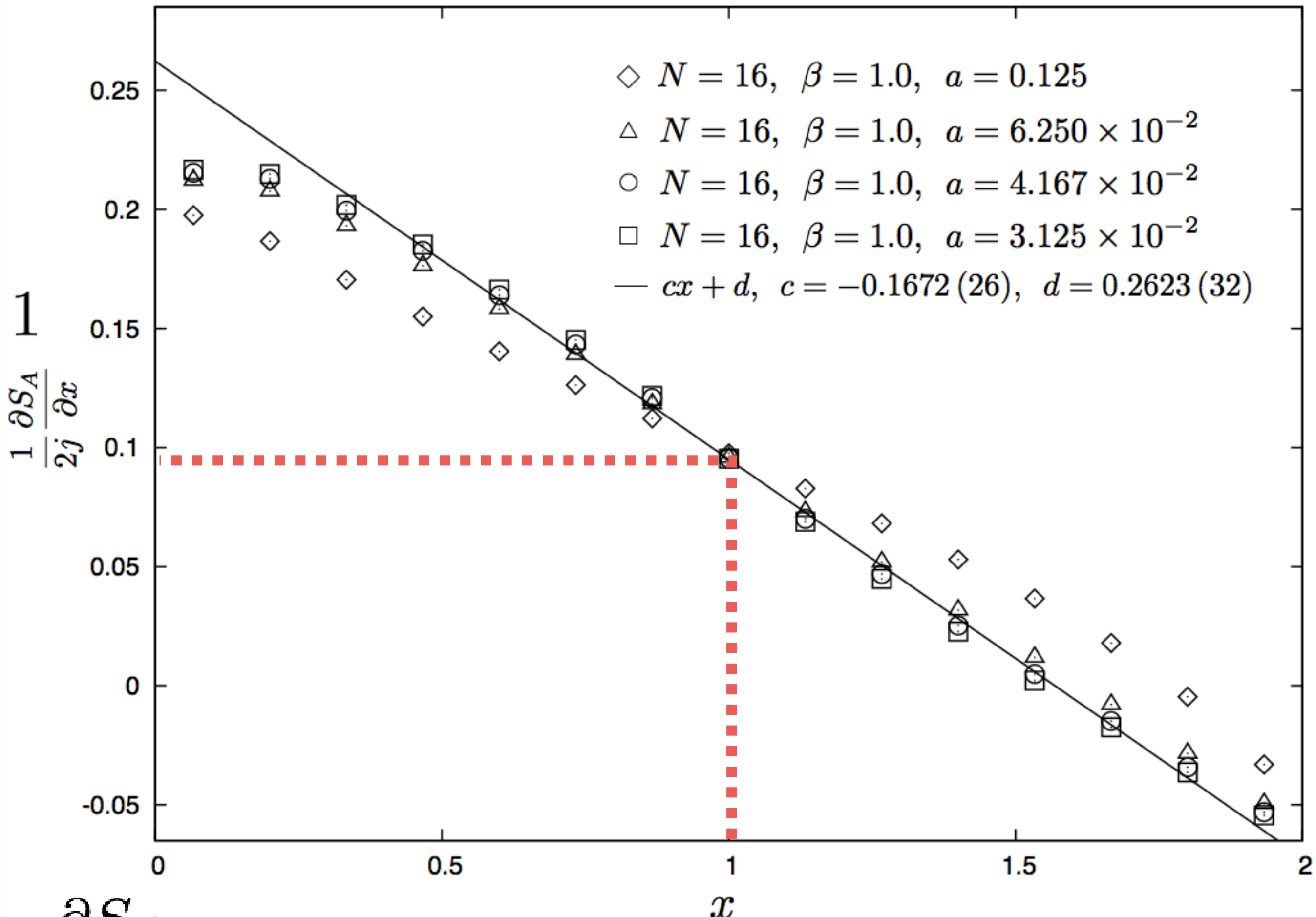
Continuum limit in the time direction at finite temperature

$\lambda = 0$

$N = 16$

$\beta = 1.0$

$N = 2j + 1$



$$\frac{\partial S_A}{\partial x}(x) = -\frac{\partial S_A}{\partial x}(2 - x)$$

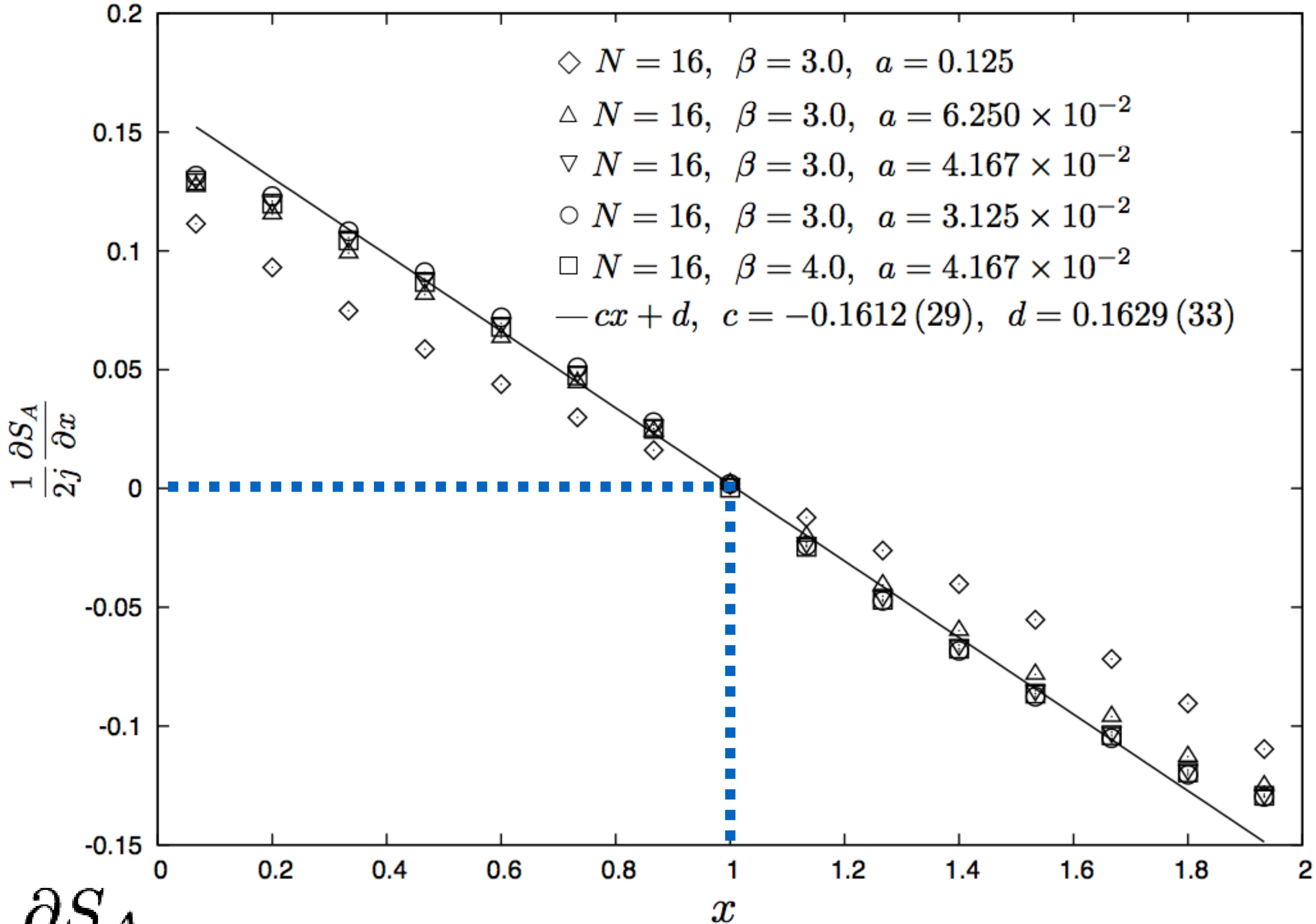
the value at $x = 1.0$ is subtracted

Continuum limit in the time direction at low temperature

$\lambda = 0$

$N = 16$

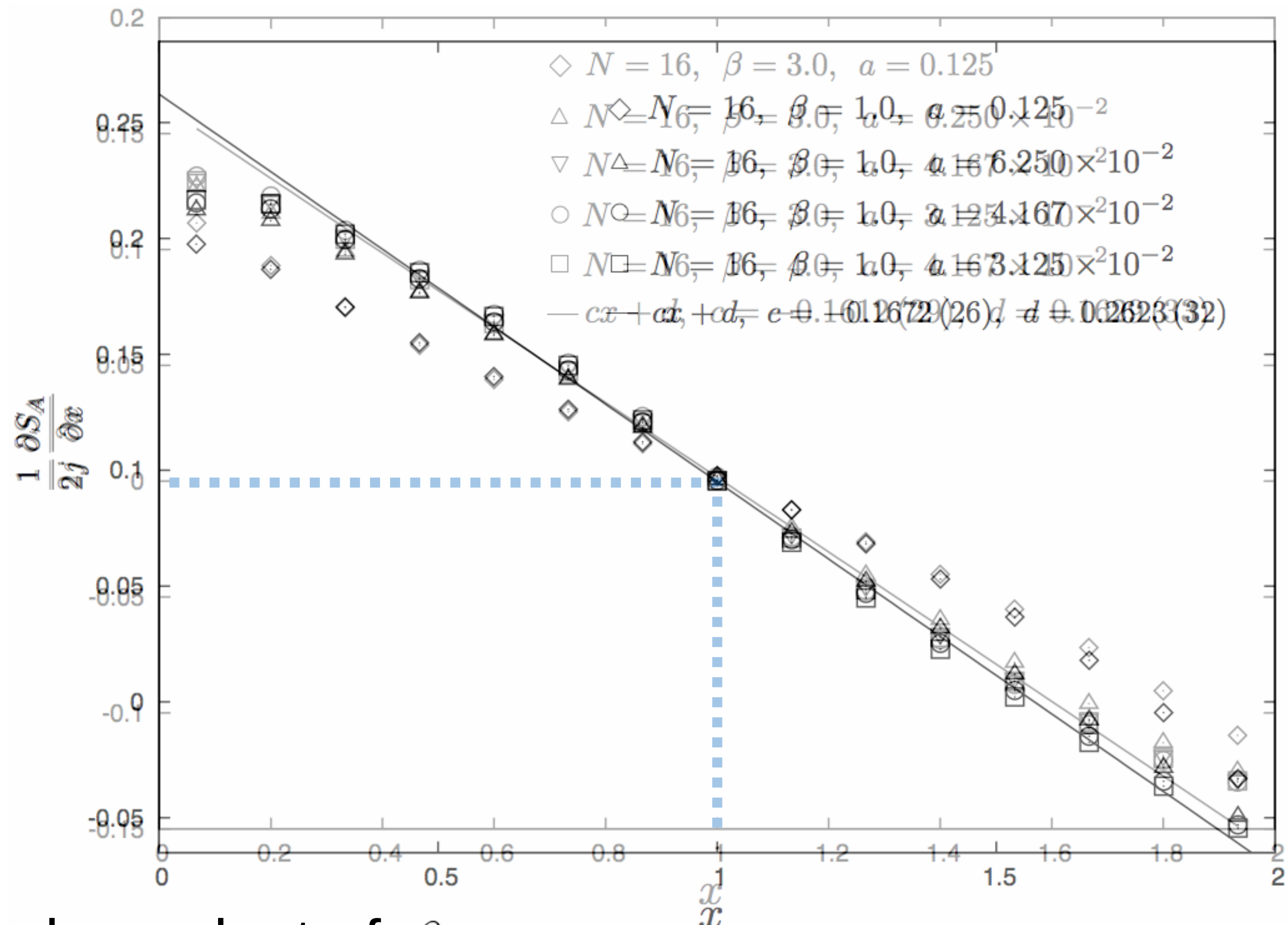
$\beta = 3.0,\ 4.0$



$$\frac{\partial S_A}{\partial x}(x) = -\frac{\partial S_A}{\partial x}(2-x)$$

Finite temperature effect

$\lambda = 0$



slope is independent of β

finite temperature effect is proportional to x (volume)

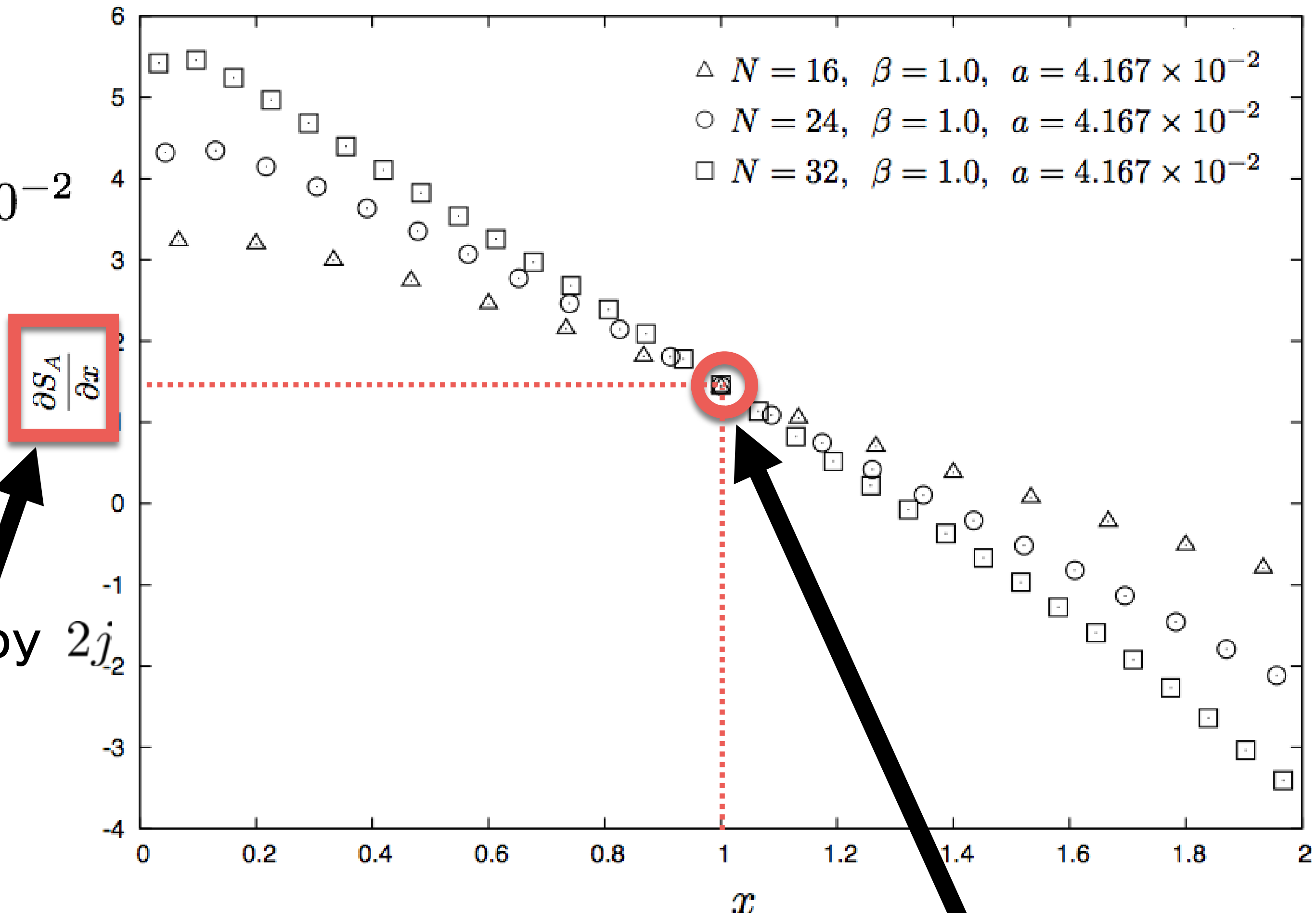
Continuum limit on the fuzzy sphere at finite temperature

$\lambda = 0$

$\beta = 1.0$

$a = 4.167 \times 10^{-2}$

$N = 2j + 1$



not divided by $2j$

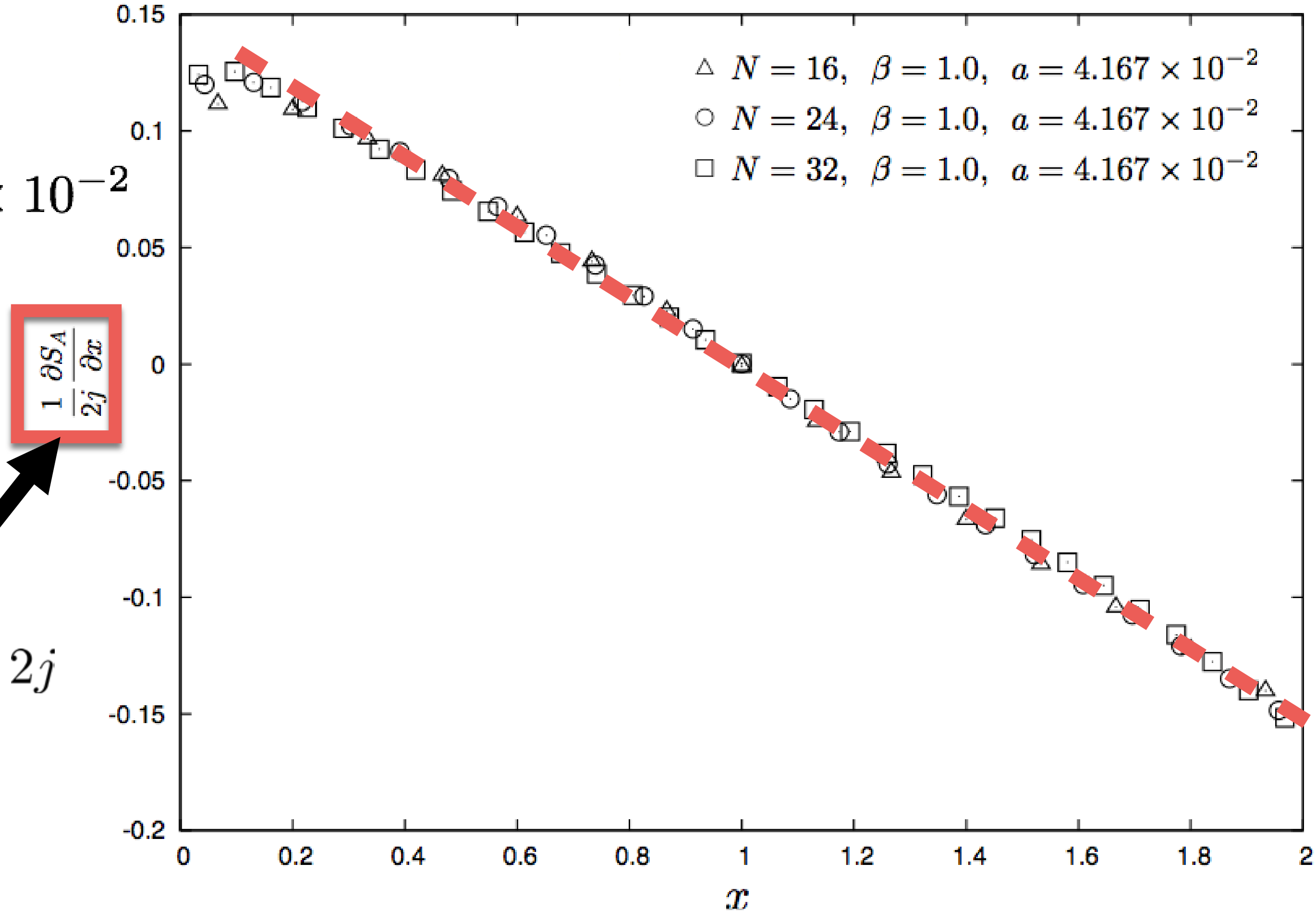
subract this value from the data

N dependence

$\lambda = 0$

$\beta = 1.0$

$a = 4.167 \times 10^{-2}$



divided by $2j$

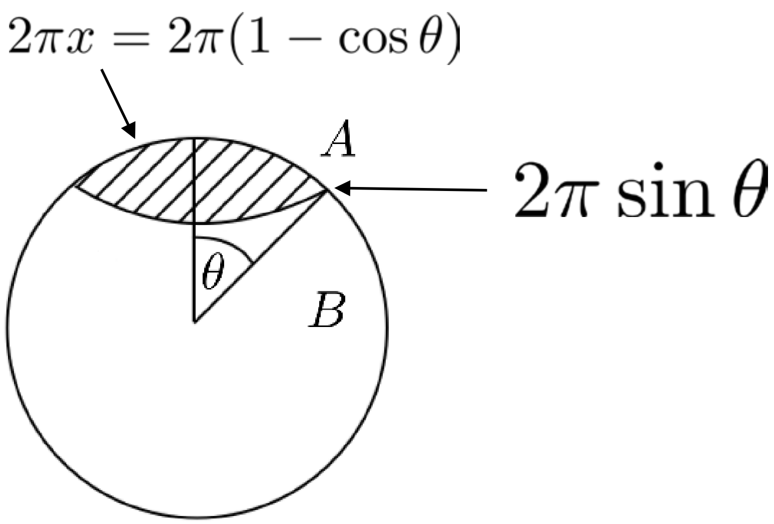
finite temperature effect is proportional to x and independent of N

S_A is proportional to N in the low temperature limit $(\beta \to \infty)$

EE in free theory

- at zero temperature ($\beta \to \infty$ limit) $N = 2j + 1$

$$\frac{1}{2j}\frac{\partial S_A}{\partial x} = c(1-x) \Longrightarrow S_A \propto N(2x - x^2) = N\sin^2\theta$$



$\propto$ (area of bndry b/w A & B)^2

differs from naïve
'area law' $\propto \sin\theta$

- finite temperature effect

$$\frac{\partial S_A}{\partial x} = g \Longrightarrow S_A = gx \quad \propto \text{ (volume of A)}$$

Results for interacting theory

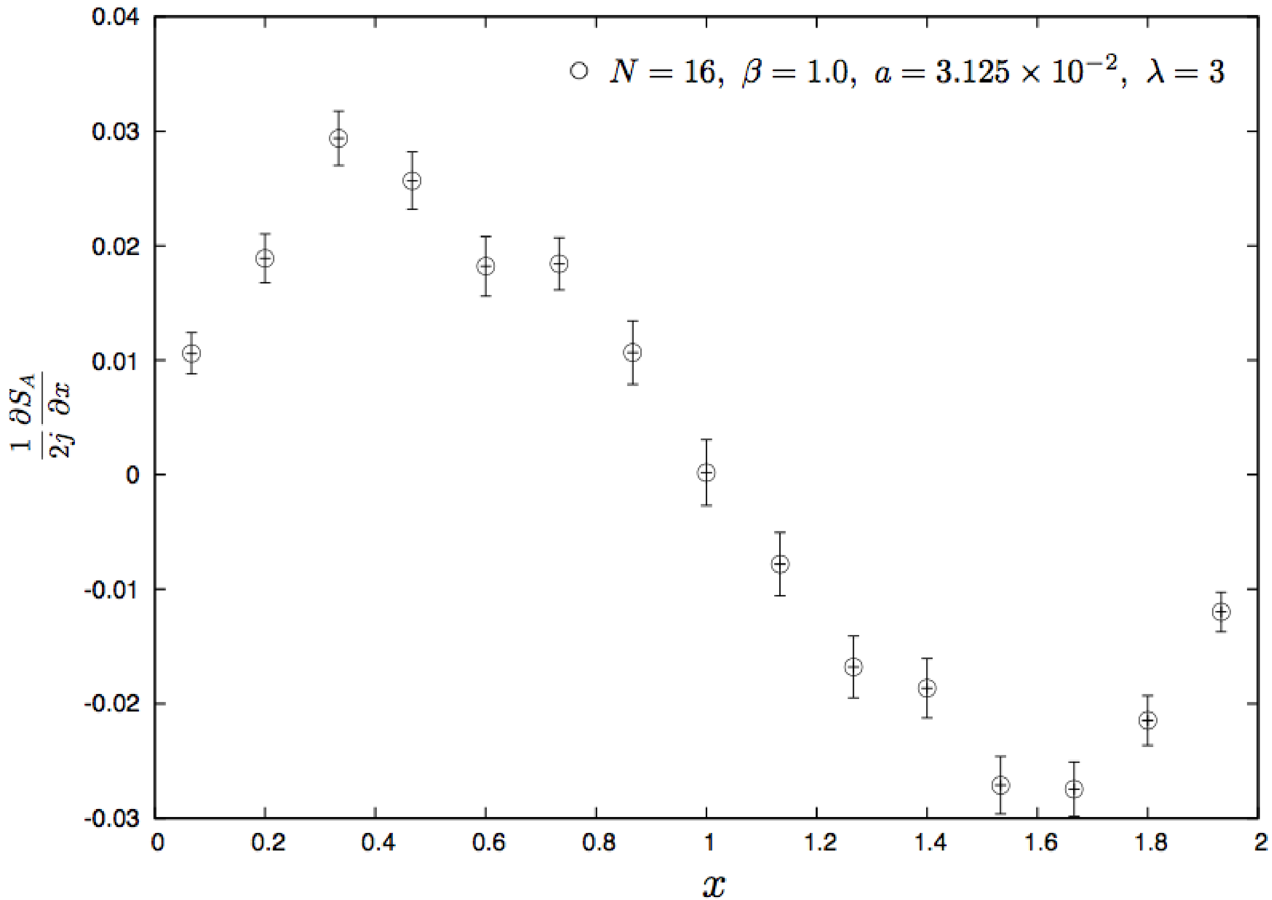
Interacting case 1

$\lambda = 3$

$N = 16$

$a = 3.125 \times 10^{-2}$

$\beta = 1.0$



different behavior from $\lambda = 0$

finite temperature effect ∝ (volume of A)

$$\frac{\partial S_A}{\partial x}(x) = -\frac{\partial S_A}{\partial x}(2 - x)$$

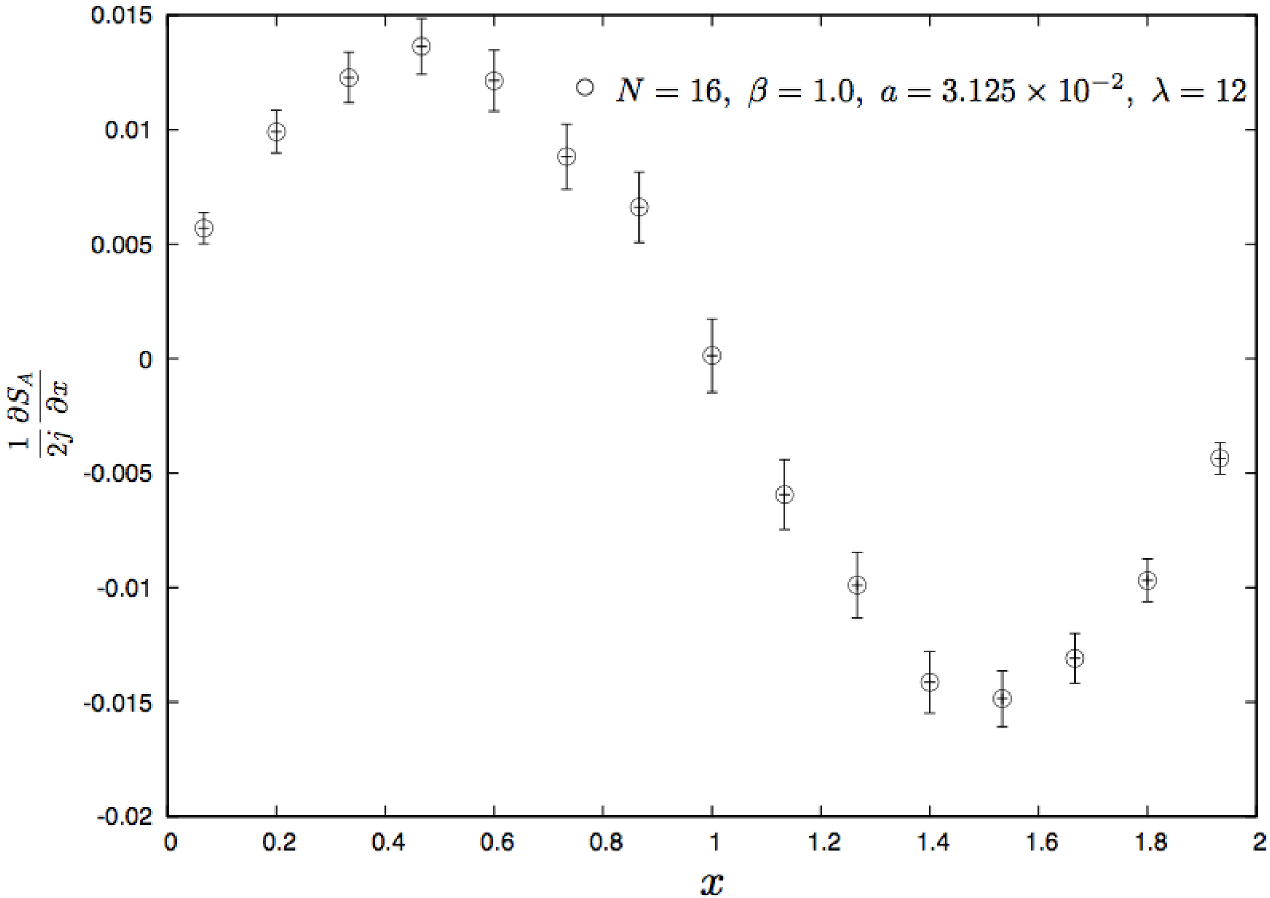
Interacting case 2

$\lambda = 12$

$N = 16$

$a = 3.125 \times 10^{-2}$

$\beta = 1.0$



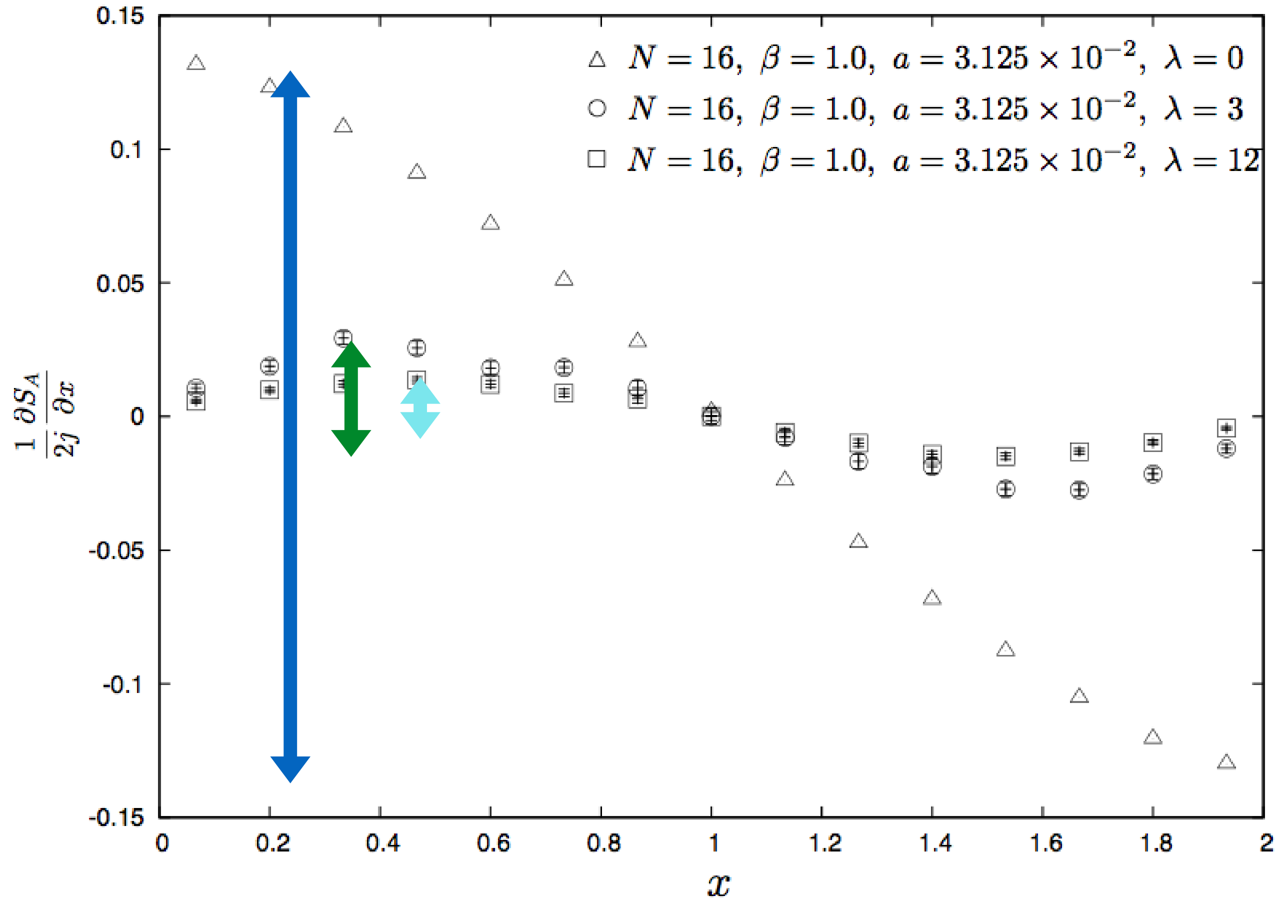
different behavior from $\lambda = 0$

finite temperature effect ∝ (volume of A)

$$\frac{\partial S_A}{\partial x}(x) = -\frac{\partial S_A}{\partial x}(2 - x)$$

Comparison with different lamdbd

$$N = 16$$
$$a = 3.125 \times 10^{-2}$$
$$\beta = 1.0$$



magnitude for $\lambda \neq 0$ is quite small compared to that for $\lambda = 0$

Conclusion and discussion

Conclusion

- free theory at zero temperature

Karczmarek-Sabella-Garnier ('13)
Sabella-Garnier ('14)

EE $\propto$ N (area of bndry b/w A & B)^2

- interacting theory at zero temperature

EE behaves differently from free theory
magnitude of EE is quite small due to non-locality and/or strong coupling

- finite temperature effect

EE $\propto$ (volume of A)
independent of N (in free theory)

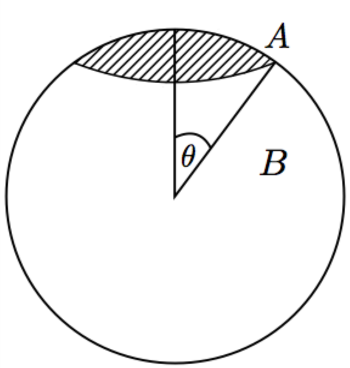
Reason for EE ∝(area)^2 in free theory

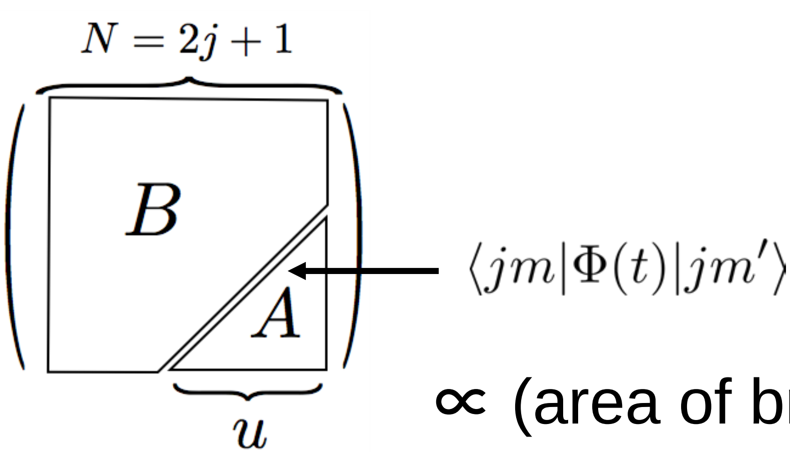
leading term in EE is dependent on UV cutoff (proportional to N)

⇒ we have to go back to the regularized theory (matrix QM)

$$S_{NC} = \frac{R^2}{2j+1}\int_0^\beta dt\ \mathrm{tr}\left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2R^2}[L_i,\Phi]^2 + \frac{\mu^2}{2}\Phi^2\right) \quad (\lambda = 0)$$

L_i are tri-diagonal ⇒ S_{NC} is local w.r.t. matrix elements



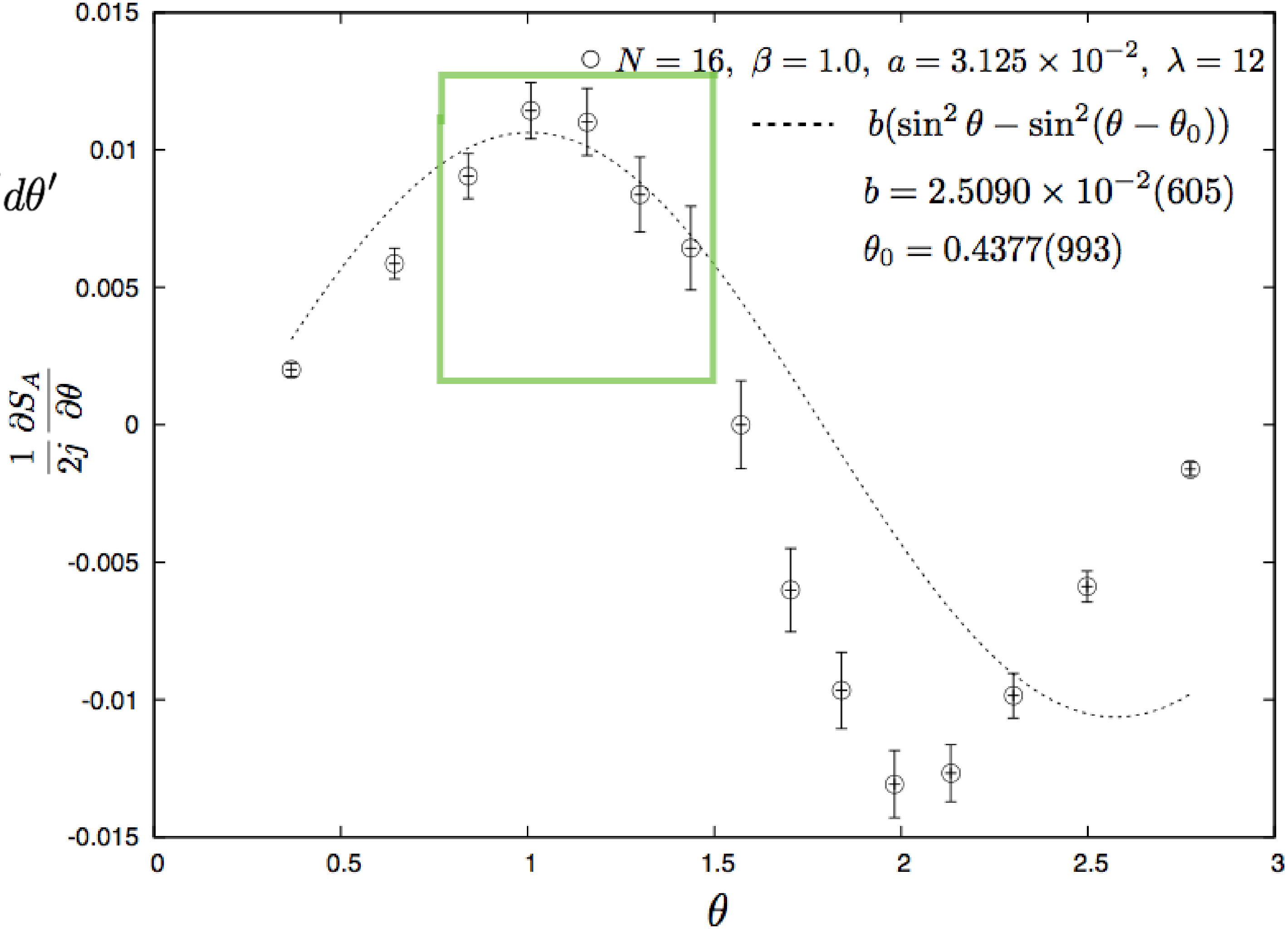


∝ (area of bndry b/w A & B)^2

Fitting for interacting case (preliminary)

'volume law'

$$S_A \propto \int_{\theta-\theta_0}^{\theta} \sin^2\theta' d\theta'$$



Outlook

- behavior of EE in interacting theory
  ~ simulation at larger N
- mutual information ~ independent of cutoff
  MI behaves in the same manner as in local field theory in the case of free theory at zero temperature
  In interacting theory? Sabella-Garnier ('14)
- planar limit in which non-commutative effect should not contribute
- gravity dual of susy gauge theory on the fuzzy sphere
  no UV/IR anomaly? Lin-Maldacena ('05)
  Cf.) Holographic EE for NCGT in R^4
  Karczmarek-Rabideau ('13)